

The role of cooperation in spatially explicit economical systems

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Abstract This paper is concerned with a model in econophysics, the subfield of statistical physics that applies concepts from traditional physics to economics. In our model, economical agents are represented by the vertices of a connected graph and are characterized by the number of coins they possess. Agents independently spend one coin at rate one for their basic need, earn one coin at a rate chosen independently from a fixed distribution ϕ and exchange money at rate μ with one of their nearest neighbors, with the richest neighbor giving one coin to the other neighbor. If an agent needs to spend one coin when her fortune is at zero, she dies, i.e., the corresponding vertex is removed from the graph. Our first results focus on the two extreme cases of lack of cooperation $\mu = 0$ and perfect cooperation $\mu = \infty$ for finite connected graphs. These results suggest that, when overall the agents earn more than they spend, cooperation is beneficial for the survival of the population, whereas when overall the agents earn less than they spend, cooperation becomes detrimental. The infinite one-dimensional system is also studied. In this case, when the agents earn less than they spend in average, the density of agents that die eventually is bounded from below by a positive constant that does not depend on the initial number of coins per agent or the level of cooperation.

1. Introduction

Models in econophysics typically consist of large systems of economical agents who earn, spend and exchange money. For a review of such models, we refer the reader to [7]. These models so far have mainly been studied by statistical physicists. From a mathematical point of view, they fall into the category of stochastic processes known as interacting particle systems [4, 6]. The most basic model in econophysics has been studied in [3] based on numerical simulations but was also considered earlier in [1, 2]. This model consists of a system of n interacting economical agents that are characterized by the number of dollars they possess, and evolves as follows: at each time step, an agent chosen uniformly at random gives one dollar to another agent again chosen uniformly at random, unless the first agent has no money in which case nothing happens. The main conjecture about this model is that, when the number of agents and the money temperature, defined as the average amount of money per agent, are both large, the limiting distribution of money is well approximated by the exponential distribution with parameter the money temperature.

Spatially explicit versions of this model where agents are located on the vertices of a finite connected graph and can only exchange money with their nearest neighbors have been recently introduced and studied analytically in [5]. The non-spatial model considered in [3] is simply obtained by assuming that the underlying graph is the complete graph with n vertices. It is proved in [5] that the conjecture in [3] is indeed correct and in fact holds for all spatially explicit versions, not only the process on the complete graph.

In this paper, we study variants of the spatially explicit models [5] where agents also earn money, spend money and die if they run out of money. In addition, we assume that the exchange of

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money occurs in a cooperative setting, meaning that the flow of money is exclusively directed from “rich” agents to “poor” agents. We also follow the framework of interacting particle systems [6] by assuming that the process evolves in continuous rather than in discrete time. This approach will allow us to define the system on infinite graphs using an idea of Harris [4] that consists in constructing the process from a collection of independent Poisson processes.

Model description. To define our spatial model formally, we let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite or infinite locally finite connected graph. Each vertex represents an economical agent who is either alive and characterized by the amount of money she possesses, or dead. To fix the ideas, we assume that the amount of money agents who are alive possess is a nonnegative integer representing a number of credits or coins, while we use the state -1 for dead agents. In particular, the state of the system at time t is a spatial configuration

$$\xi_t : \mathcal{V} \longrightarrow \{-1, 0, 1, 2, \dots\}$$

with the value of $\xi_t(x)$ indicating that agent x is dead or representing the number of coins this agent possesses when she is alive. To define the evolution rules, we attach to each vertex $x \in \mathcal{V}$ a random variable ϕ_x chosen independently from a fixed distribution ϕ . The individual at vertex x earns one coin at rate ϕ_x and, to ensure her survival, spends one coin at rate one. The population is also characterized by its level of cooperation which is measured using a nonnegative parameter μ as follows: nearest neighbors that are alive interact at rate μ and, in case one neighbor has at least two more coins than the other neighbor, she gives one coin to the other neighbor. In particular, the “richest” agent before the interaction does not give any coin if this makes her “poorer” than her neighbor. Finally, if an individual has zero coin at the time she needs to spend one coin then she dies and the corresponding vertex is removed from the graph. To describe the dynamics formally, for each spatial configuration ξ , we let

$$\begin{aligned} \textbf{spending} \quad \xi_x^-(z) &= \xi(z) - \mathbf{1}\{z = x\} \quad \text{for all } z \in \mathcal{V} \\ \textbf{earning} \quad \xi_x^+(z) &= \xi(z) + \mathbf{1}\{z = x\} \quad \text{for all } z \in \mathcal{V} \end{aligned}$$

be the configurations obtained respectively by removing/adding one coin at vertex x . Also, for each edge $(x, y) \in \mathcal{E}$ of the network of interactions, we let

$$\begin{aligned} \textbf{cooperation} \quad \xi_{(x,y)}(z) &= \xi(z) + \mathbf{1}\{\xi(x) < \xi(y) - 1\}(\mathbf{1}\{z = x\} - \mathbf{1}\{z = y\}) \\ &\quad + \mathbf{1}\{\xi(y) < \xi(x) - 1\}(\mathbf{1}\{z = y\} - \mathbf{1}\{z = x\}) \end{aligned}$$

be the configuration obtained by moving one coin from the richer to the poorer vertex if the two vertices are at least two coins apart. The dynamics of the system is then described by the Markov generator L defined on the set of cylinder functions by

$$\begin{aligned} Lf(\xi) &= \sum_{x \in \mathcal{V}} (f(\xi_x^-) - f(\xi)) \mathbf{1}\{\xi(x) \neq -1\} \\ &\quad + \sum_{x \in \mathcal{V}} \phi_x (f(\xi_x^+) - f(\xi)) \mathbf{1}\{\xi(x) \neq -1\} \\ &\quad + \sum_{(x,y) \in \mathcal{E}} \mu (f(\xi_{(x,y)}) - f(\xi)) \mathbf{1}\{\xi(x) \neq -1, \xi(y) \neq -1\}. \end{aligned}$$

The first sum describes the rate at which vertices spend one coin, the second sum the rate at which they earn one coin, and the third sum the rate at which neighbors exchange one coin. As previously mentioned, the process is well defined on locally finite graphs, including infinite graphs, and can be constructed from a collection of independent Poisson processes. More precisely,

- for all $x \in \mathcal{V}$, let $N_t^-(x)$ be a Poisson process with intensity one,
- for all $x \in \mathcal{V}$, let $N_t^+(x)$ be a Poisson process with intensity ϕ_x ,
- for all $(x, y) \in \mathcal{E}$, let $N_t(x, y)$ be a Poisson process with intensity μ .

We further assume that these processes are independent. This implies that, with probability one, the arrival times are all distinct. A general result due to Harris [4] then shows that the process can be constructed using the following rules:

- At the arrival times of the Poisson process $N_t^-(x)$, we take one coin from the individual at vertex x if this individual is still alive.
- At the arrival times of the Poisson process $N_t^+(x)$, we give one coin to the individual at vertex x if this individual is still alive.
- At the arrival times of $N_t^+(x, y)$, we move one coin from x to y if x has at least two more coins than y or one coin from y to x if y has at least two more coins than x .

Main results. To begin with, we compare the two processes with the same earning rates ϕ_z in the absence of cooperation $\mu = 0$ and in the presence of perfect cooperation $\mu = \infty$ on finite connected graphs to understand whether cooperation is beneficial or detrimental for survival. Our first results look at the probability of global survival that we define as

$$p_\mu(c, (\phi_z)) = P(\xi_t(z) \neq -1 \text{ for all } (z, t) \in \mathcal{V} \times \mathbb{R}_+ \mid \xi_0 \equiv c)$$

where c refers to the common initial number of coins per agent and where the earning rates ϕ_z are independent realizations of the distribution ϕ for all $z \in \mathcal{V}$. Estimates for the probability of global survival can be expressed in terms of the two key quantities

$$\mathcal{D} = \max_{x \in \mathcal{V}} \sum_{z \in \mathcal{V}} d(x, z) \quad \text{and} \quad \Phi = (1/n) \sum_{z \in \mathcal{V}} \phi_z \quad (1)$$

where d refers to the graph distance and n to the population size. Using that, as long as all the agents are alive, the total number of coins on the graph behaves like a random walk that increases at rate $n\Phi$ and decreases at rate n together with the fact that nearest neighbors are at most one coin apart in the presence of perfect cooperation, we get the following theorem.

Theorem 1 – In the presence of perfect cooperation $\mu = \infty$,

$$p_\infty(c, (\phi_z)) \geq \max(0, 1 - \Phi^{-(nc - \mathcal{D} + 1)}).$$

The proof relies, among other things, on an application of the optional stopping theorem for martingales. The inequality in the statement turns out to be an equality when $n = 1$. In particular, since the system in the absence of cooperation behaves like n independent copies of a one-person system, the theorem also gives the probability of global survival when $\mu = 0$. Using this and some basic algebra, it can be proved that, when $\Phi > 1$ and c is large, the probability of global survival is larger in the presence of perfect cooperation than in the absence of cooperation.

Theorem 2 – Assume that $\Phi > 1$. Then, there exists $c_0 < \infty$ such that,

$$\begin{aligned} p_0(c, (\phi_z)) &= \prod_{z \in \mathcal{V}} \max(0, 1 - \phi_z^{-(c+1)}) \\ &\leq \max(0, 1 - \Phi^{-(nc - \mathcal{D} + 1)}) \leq p_\infty(c, (\phi_z)) \quad \text{for all } c \geq c_0. \end{aligned}$$

More generally, we conjecture that, when $\Phi > 1$, i.e., when overall the agents earn more than they spend, the probability of global survival is larger in the presence of perfect cooperation than in the absence of cooperation regardless of the initial value c .

We now focus on the two-person system: we set $\mathcal{V} = \{x, y\}$ and assume that vertices x and y are connected by an edge. In this case, Theorem 1 implies that when

$$\Phi = (1/2)(\phi_x + \phi_y) > 1 \quad \text{and} \quad \phi_x < 1 < \phi_y$$

global survival is possible in the presence of perfect cooperation whereas individual x dies almost surely in the absence of cooperation, showing again that cooperation is beneficial. Cooperation, however, becomes detrimental when

$$\Phi = (1/2)(\phi_x + \phi_y) < 1 \quad \text{and} \quad \phi_x < 1 < \phi_y.$$

In this case, regardless of the level of cooperation μ , global survival is not possible so, to measure the effect of cooperation, we study instead

$$E_\mu(c, (\phi_z)) = E(\text{card}\{z \in \mathcal{V} : \xi_t(z) \neq -1 \text{ for all } t \in \mathbb{R}_+ \mid \xi_0 \equiv c\}),$$

the expected number of individuals that live forever. Due to perfect cooperation and the fact that individual x dies almost surely, it can be proved that the last time both individuals each have one coin is almost surely finite and that, between this time and the first time one of the two individuals dies, the process behaves according to a certain seven-state Markov chain. Using a first-step analysis to study this Markov chain and part of the proof of Theorem 1, the expected value of the number of individuals that live forever can be computed explicitly.

Theorem 3 – Assume that $\mathcal{V} = \mathcal{E} = \{x, y\}$ and that

$$\Phi = (1/2)(\phi_x + \phi_y) < 1 \quad \text{and} \quad \phi_x < 1 < \phi_y.$$

Then, letting $\Psi = 8 + 2\phi_x + 2\phi_y$, for all $c \geq 1$,

$$\begin{aligned} E_\infty(c, \phi_x, \phi_y) &= \left(\frac{2}{\Psi}\right)\left(1 - \frac{1}{\phi_y}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right)\left(1 - \left(\frac{1}{\phi_y}\right)^2\right) \\ &< 1 - \left(\frac{1}{\phi_y}\right)^{c+1} = E_0(c, \phi_x, \phi_y). \end{aligned}$$

Our approach to prove this result works in theory for all complete graphs, but becomes computationally intractable even with only three vertices. More generally, we conjecture that, at least on the complete graph and when $\Phi < 1$, i.e., when overall the agents earn less than they spend, the expected number of individuals that live forever is larger in the absence of cooperation than in the presence of perfect cooperation. In a nutshell, we conjecture that cooperation is beneficial for populations that are “productive” but detrimental for populations that are not.

Finally, we look at the infinite system in one dimension: the underlying graph is represented by the integers with each integer being connected to its predecessor and to its successor. In this case, the process is more difficult to study because the graph is infinite. The next result shows that, when the expected value of ϕ is less than one, the density of individuals who die eventually in the infinite one-dimensional system is bounded from below by a positive constant that does not depend on the initial number of coins per agent.

Theorem 4 – Assume that $E(\phi) < 1$. Then,

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbf{1}\{\xi_t(z) = -1 \text{ for some } t\} = l$$

where $l > 0$ does not depend on the initial fortune c per vertex.

To prove this result, we first identify a collection of events that ensure that a given agent dies before time one. This, together with the ergodic theorem, implies that the density of agents that die before time one is positive. This density, however, depends *a priori* on the initial fortune. Then, we define a sink as a vertex such that the agents in any finite interval that contains this vertex earn overall less than they spend. The law of large numbers implies that the density of sinks is bounded from below by a constant that does not depend on the initial fortune. Using finally that, at time one, each sink is located between two agents who already died, we use a recursive argument to prove that each sink dies eventually. In conclusion, the density of individuals who die eventually is bounded from below by the density of sinks which, in turn, is bounded from below by a positive constant that does not depend on the initial fortune. This gives the result.

The proof of Theorem 4 also suggests that, when the expected value of ϕ is larger than one, the density of agents who live forever can be made arbitrarily close to one by choosing the initial fortune c large enough. The proof of this result, however, requires additional arguments that we were not able to make rigorous.

2. Proof of Theorems 1 and 2

In this section, we start by collecting some preliminary results about martingales that will be used later to prove the first two theorems. The first step is to estimate probabilities related to the continuous-time Markov chain (W_t) with transition rates

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(W_{t+\epsilon} = W_t + 1) &= \sum_{z \in \mathcal{V}} \phi_z \\ \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(W_{t+\epsilon} = W_t - 1) &= \text{card}(\mathcal{V}) = n. \end{aligned} \tag{2}$$

Recall from (1) that $\Phi = (1/n) \sum_{z \in \mathcal{V}} \phi_z$.

Lemma 5 – The process (Φ^{-W_t}) is a martingale.

Proof. Letting (\mathcal{F}_t) denote the natural filtration of the process (W_t) , and recalling the expression of its transition rates in (2), we get

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \epsilon^{-1} E(\Phi^{-W_{t+\epsilon}} - \Phi^{-W_t} \mid \mathcal{F}_t) \\ &= (\sum_{z \in \mathcal{V}} \phi_z)(\Phi^{-(W_t+1)} - \Phi^{-W_t}) + n(\Phi^{-(W_t-1)} - \Phi^{-W_t}) \\ &= n\Phi(\Phi^{-(W_t+1)} - \Phi^{-W_t}) + n(\Phi^{-(W_t-1)} - \Phi^{-W_t}) \\ &= n(\Phi^{-W_t} - \Phi^{-(W_t-1)}) + n(\Phi^{-(W_t-1)} - \Phi^{-W_t}) = 0, \end{aligned}$$

which shows that (Φ^{-W_t}) is a martingale. \square

To state our next results, we define the stopping times

$$T_i = \inf \{t : W_t = i\} \quad \text{for all } i \in \mathbb{Z}.$$

Lemma 6 – Assume that $M \leq nc \leq N$ and $\Phi \neq 1$. Then,

$$p(M, N) = P(T_N < T_M \mid W_0 = nc) = \frac{1 - \Phi^{-(nc-M)}}{1 - \Phi^{-(N-M)}}.$$

Proof. Since (Φ^{-W_t}) is a martingale and the process stopped at time $T = \min(T_M, T_N)$ is bounded, we may apply the optional stopping theorem to get

$$E(\Phi^{-W_T}) = E(\Phi^{-W_0}) = \Phi^{-nc}. \quad (3)$$

Note also that

$$\begin{aligned} E(\Phi^{-W_T}) &= E(\Phi^{-W_T} \mid T = T_M)(1 - p(M, N)) + E(\Phi^{-W_T} \mid T = T_N)p(M, N) \\ &= \Phi^{-M}(1 - p(M, N)) + \Phi^{-N}p(M, N) \\ &= (\Phi^{-N} - \Phi^{-M})p(M, N) + \Phi^{-M}. \end{aligned} \quad (4)$$

Combining (3) and (4), we conclude that

$$p(M, N) = \frac{\Phi^{-nc} - \Phi^{-M}}{\Phi^{-N} - \Phi^{-M}} = \frac{1 - \Phi^{-(nc-M)}}{1 - \Phi^{-(N-M)}}.$$

This completes the proof. \square

Lemma 7 – For all $M \leq nc$ and all $\Phi > 0$,

$$q(M) = P(T_M = \infty \mid W_0 = nc) = \max(0, 1 - \Phi^{-(nc-M)}).$$

Proof. We distinguish three cases depending on the value of Φ .

- When $\Phi = 1$, the process (W_t) is the one-dimensional symmetric random walk which is known to be recurrent. This gives the probability $P(T_M = \infty) = 0$.
- When $\Phi < 1$, the law of large numbers implies that $W_t \rightarrow -\infty$ almost surely. In particular, the stopping time T_M is again almost surely finite and the probability $q(M) = 0$.
- When $\Phi > 1$, the law of large numbers now gives $W_t \rightarrow \infty$ so

$$\{T_M = \infty\} = \{T_N < T_M \text{ for all } N \geq nc\} \quad \text{almost surely.}$$

Since in addition we have the inclusions

$$\{T_{N+1} < T_M\} \subset \{T_N < T_M\} \quad \text{for all } N \geq nc,$$

by monotone convergence and Lemma 6, we get

$$\begin{aligned} q(M) &= P(T_N < T_M \text{ for all } N \geq nc \mid W_0 = nc) \\ &= P(\lim_{N \rightarrow \infty} \{T_N < T_M\} \mid W_0 = nc) \\ &= \lim_{N \rightarrow \infty} P(T_N < T_M \mid W_0 = nc) = 1 - \Phi^{-(nc-M)}. \end{aligned}$$

Observing also that $1 - \Phi^{-(nc-M)} \leq 0$ if and only if $\Phi \leq 1$ gives the result. \square

Lemma 7 is the main ingredient to prove Theorem 1. To see the connection between the previous martingale results and the economical system, define

$$\tau = \inf \{t : \xi_t(x) = -1 \text{ for some } x \in \mathcal{V}\} \quad \text{and} \quad Z_t = \sum_{z \in \mathcal{V}} \xi_t(z)$$

and observe that, before time τ , the individual at z is alive, earns one coin at rate ϕ_z and spends one coin at rate one, therefore

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(Z_{t+\epsilon} = Z_t + 1 \mid \tau > t) &= \sum_{z \in \mathcal{V}} \phi_z \\ \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P(Z_{t+\epsilon} = Z_t - 1 \mid \tau > t) &= \text{card}(\mathcal{V}) = n. \end{aligned}$$

In other words, by time τ , the total number of coins behaves like the Markov chain (W_t) . Using this and the previous lemma, we can now prove the theorem.

Proof of Theorem 1. In the limiting case $\mu = \infty$ and as long as all the individuals are alive, each time an individual has at least two more coins than one of her neighbors, this individual instantaneously gives a coin to one of her poorest neighbors, therefore

$$|\xi_t(x) - \xi_t(y)| \leq 1 \quad \text{for all } (x, y) \in \mathcal{E} \text{ and } t < \tau.$$

Now, letting $x, y \in \mathcal{V}$ be arbitrary, there exist

$$z_0 = x, z_1, \dots, z_d = y \in \mathcal{V} \quad \text{such that} \quad (z_i, z_{i+1}) \in \mathcal{E} \quad \text{for all } i = 0, 1, \dots, d-1$$

where $d = d(x, y)$. In particular, the triangle inequality implies that

$$\begin{aligned} |\xi_t(x) - \xi_t(y)| &\leq |\xi_t(z_0) - \xi_t(z_1)| + \dots + |\xi_t(z_{d-1}) - \xi_t(z_d)| \\ &\leq d = d(x, y) \end{aligned} \tag{5}$$

for all $t < \tau$. Now, on the event that $\tau < \infty$, just before that time, there is at least one vertex, say x , with zero coin, while the other vertices have a positive fortune. This, together with (5), implies that the total number of coins satisfies

$$Z_{\tau-} = \sum_{z \in \mathcal{V}} \xi_{\tau-}(z) = \sum_{z \in \mathcal{V}} |\xi_{\tau-}(x) - \xi_{\tau-}(z)| \leq \sum_{z \in \mathcal{V}} d(x, z).$$

Taking the maximum over all possible configurations gives

$$Z_{\tau-} \leq \max_{x \in \mathcal{V}} \sum_{z \in \mathcal{V}} d(x, z) = \mathcal{D}.$$

Finally, using Lemma 7 and observing that all the individuals survive if and only if $\tau = \infty$ give the following lower bound for the probability of global survival

$$\begin{aligned} p_\infty(c, (\phi_z)) &= P(\tau = \infty \mid \xi_0(z) = c \text{ for all } z \in \mathcal{V}) \\ &\geq P(Z_t \geq \mathcal{D} \text{ for all } t \mid \xi_0(z) = c \text{ for all } z \in \mathcal{V}) \\ &= P(W_t > \mathcal{D} - 1 \text{ for all } t \mid W_0 = nc) \\ &= P(T_{\mathcal{D}-1} = \infty \mid W_0 = nc) = q(\mathcal{D} - 1) \\ &= \max(0, 1 - \Phi^{-(nc - \mathcal{D} + 1)}). \end{aligned}$$

This completes the proof of the theorem. \square

Using Lemma 7 and Theorem 1, we can now prove Theorem 2.

Proof of Theorem 2. It follows from Lemma 7 that, in the presence of only one vertex, say x , the probability of survival is given by

$$p_0(c, \phi_x) = q(-1) = \max(0, 1 - \phi_x^{-(c+1)}).$$

Since in the absence of cooperation $\mu = 0$, the system with n individuals consists of n independent copies of a one-person system, we get

$$p_0(c, (\phi_z)) = \prod_{z \in \mathcal{V}} p_0(c, \phi_z) = \prod_{z \in \mathcal{V}} \max(0, 1 - \phi_z^{-(c+1)}).$$

It directly follows that

$$p_0(c, (\phi_z)) = 0 \quad \text{when} \quad \phi_z \leq 1 \quad \text{for some} \quad z \in \mathcal{V}$$

so the inequality to be proved is obvious in this case. Assume now that $\phi_z > 1$ for all $z \in \mathcal{V}$. In this case, we have the following inequalities:

$$\begin{aligned} \log(p_0(c, (\phi_z))) &= \sum_{z \in \mathcal{V}} \log(1 - \phi_z^{-(c+1)}) \leq -\sum_{z \in \mathcal{V}} \phi_z^{-(c+1)} \\ \log(p_\infty(c, (\phi_z))) &\geq \log(1 - \Phi^{-(nc-\mathcal{D}+1)}) \geq -\Phi^{-(nc-\mathcal{D}+1)} / (1 - \Phi^{-(nc-\mathcal{D}+1)}). \end{aligned}$$

In particular, since $\Phi > 1$, for all $n \geq 2$ and c sufficiently large,

$$\begin{aligned} \log(p_\infty(c, (\phi_z))) &\geq -\Phi^{-(nc-\mathcal{D}+1)} / (1 - \Phi^{-(nc-\mathcal{D}+1)}) \\ &\geq -2 \Phi^{-(nc-\mathcal{D}+1)} \geq -2 (\min_{z \in \mathcal{V}} \phi_z)^{-(nc-\mathcal{D}+1)} \\ &\geq -(\min_{z \in \mathcal{V}} \phi_z)^{-(c+1)} \geq -\sum_{z \in \mathcal{V}} \phi_z^{-(c+1)} \\ &\geq \log(p_0(c, (\phi_z))). \end{aligned}$$

This completes the proof of the theorem. \square

3. Proof of Theorem 3

As stated in the introduction, the two-person system is simple enough that we may calculate certain probabilities by hand. Since there are only two vertices, we will call them x and y and the rates at which they earn a coin ϕ_x and ϕ_y , respectively. To simplify the notation, write

$$X_t = \xi_t(x) \quad \text{and} \quad Y_t = \xi_t(y) \quad \text{for all} \quad t \geq 0.$$

Letting $T_- = \inf \{t : \min(X_t, Y_t) = -1\}$, the process

$$\Phi^{-(X_{t \wedge T_-} + Y_{t \wedge T_-})} = \left(\frac{2}{\phi_x + \phi_y} \right)^{X_{t \wedge T_-} + Y_{t \wedge T_-}}$$

is again a martingale. Using that the individuals' fortune differ by at most one coin in the presence of perfect cooperation, and repeating the proofs of Lemmas 6 and 7, we easily show that, when both individuals start with c coins, the probability of global survival satisfies

$$\begin{aligned} p_\infty(c, \phi_x, \phi_y) &= P(\min(X_t, Y_t) \geq 0 \text{ for all } t \mid X_0 = Y_0 = c) \\ &\geq P(X_t + Y_t > 0 \text{ for all } t \mid X_0 = Y_0 = c) \\ &= \max(0, 1 - (2/(\phi_x + \phi_y))^{2c}) \end{aligned}$$

in the case of perfect cooperation. In particular, when

$$\phi_x + \phi_y > 2 \quad \text{and} \quad \phi_x < 1 < \phi_y,$$

while individual x dies almost surely in the absence of cooperation, global survival is possible in the presence of perfect cooperation, showing that cooperation is beneficial in this case. We now focus on the parameter region

$$\phi_x + \phi_y < 2 \quad \text{and} \quad \phi_x < 1 < \phi_y \tag{6}$$

and show that, in this case, cooperation is detrimental: individual x again dies almost surely while individual y is more likely to live forever in the absence of cooperation than in the presence of perfect cooperation. The probability of survival can be computed explicitly.

Using again that the individuals' fortune differ by at most one coin in the presence of perfect cooperation, together with the fact that global survival is not possible when (6) holds, implies that the stopping time T_- is almost surely finite and that

$$(X_{T_-}, Y_{T_-}) \in \{(-1, 0), (-1, 1), (0, -1), (1, -1)\}.$$

To simplify the notation, we rename these four states as well as the three adjacent states as shown in Figure 1 and define the stopping times and corresponding probabilities

$$\tau_i = \inf \{t : (X_t, Y_t) = S_i\} \quad \text{and} \quad p_i = P(T_- = \tau_i) \quad \text{for } i = 1, 2, 3, 4.$$

The probabilities p_i are computed explicitly in the next lemma.

Lemma 8 – Assume (6) and perfect cooperation. Then,

$$p_1 = p_2 = 2/\Psi \quad p_3 = \phi_x/\Psi + 1/4 \quad p_4 = \phi_y/\Psi + 1/4$$

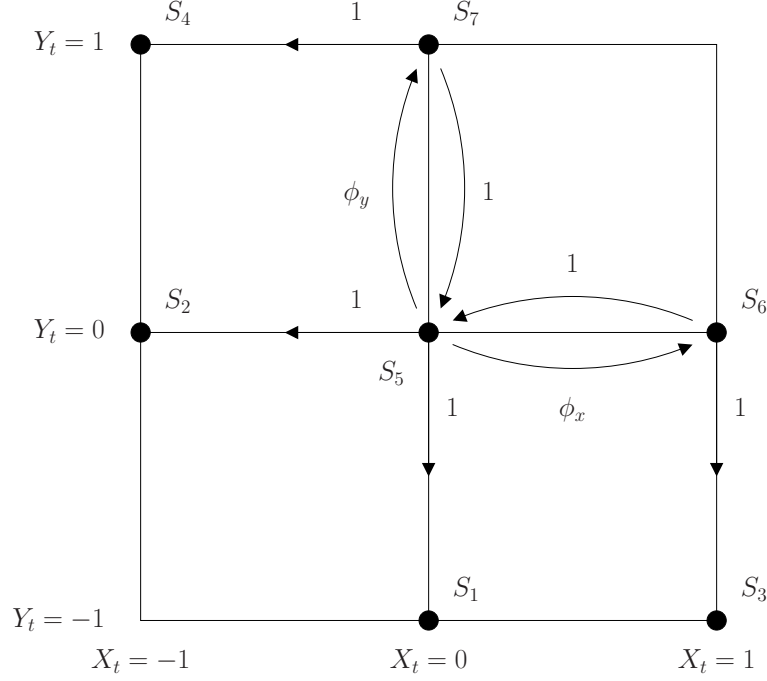
where $\Psi = 8 + 2\phi_x + 2\phi_y$.

Proof. Observe that T_- is almost surely finite when (6) holds. Since in addition the individuals' fortune differ by at most one coin before time T_- ,

$$T_+ = \sup \{t : X_t = Y_t = 1\} < \infty \quad \text{almost surely.}$$

Also, between time T_+ and time T_- , the process consists of the seven-state continuous-time Markov chain whose transition rates are indicated in Figure 1. Referring again to the picture for the name of the states, we define the conditional probabilities

$$p_{ij} = P(T_- = \tau_i \mid (X_0, Y_0) = S_j) \quad \text{for all } (i, j) \in \{1, 2, 3, 4\} \times \{5, 6, 7\}.$$

FIGURE 1. The seven states and transition rates between times T_+ and T_- .

Using a first-step analysis and looking at the probabilities at which the process starting from state S_5 jumps to each of the four adjacent states, we get

$$p_{15} = \frac{1}{2 + \phi_x + \phi_y} + \frac{\phi_x p_{16}}{2 + \phi_x + \phi_y} + \frac{\phi_y p_{17}}{2 + \phi_x + \phi_y}.$$

The same idea gives $p_{16} = p_{17} = (1/2) p_{15}$. Solving the system, we get

$$p_{15} = \frac{2}{4 + \phi_x + \phi_y} \quad \text{and} \quad p_{16} = p_{17} = \frac{1}{4 + \phi_x + \phi_y}.$$

Since in addition the first state visited after time T_+ is equally likely to be S_6 and S_7 , we conclude that the probability p_1 is given by

$$p_1 = \frac{p_{16} + p_{17}}{2} = \frac{1}{4 + \phi_x + \phi_y} = \frac{2}{\Psi}$$

which, by symmetry, is also the value of p_2 . To compute p_3 , we again use a first-step analysis to obtain a system involving the three conditional probabilities:

$$p_{35} = \frac{\phi_x p_{36}}{2 + \phi_x + \phi_y} + \frac{\phi_y p_{37}}{2 + \phi_x + \phi_y} \quad p_{36} = \frac{1}{2} + \frac{p_{35}}{2} \quad p_{37} = \frac{p_{35}}{2}.$$

Solving the system gives

$$p_{35} = \frac{\phi_x}{4 + \phi_x + \phi_y} \quad p_{36} = \frac{1}{2} + \frac{\phi_x}{8 + 2\phi_x + 2\phi_y} \quad p_{37} = \frac{\phi_x}{8 + 2\phi_x + 2\phi_y}$$

from which it follows as before that

$$p_3 = \frac{p_{36} + p_{37}}{2} = \frac{\phi_x}{8 + 2\phi_x + 2\phi_y} + \frac{1}{4} = \frac{\phi_x}{\Psi} + \frac{1}{4}.$$

By symmetry, the value of p_4 is obtained by exchanging the role of ϕ_x and ϕ_y in the previous expression, which completes the proof. \square

Using the previous lemma as well as Lemma 7 and conditioning on the first boundary state visited, we deduce that the expected number of individuals that survive in the presence of perfect cooperation, which is also the probability that y survives, is given by

$$E_\infty(c, \phi_x, \phi_y) = p_2 p_0(0, \phi_y) + p_4 p_0(1, \phi_y) = \left(\frac{2}{\Psi}\right) \left(1 - \frac{1}{\phi_y}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right) \left(1 - \left(\frac{1}{\phi_y}\right)^2\right).$$

Since in addition

$$1 - \frac{1}{\phi_y} < 1 - \left(\frac{1}{\phi_y}\right)^2 \leq 1 - \left(\frac{1}{\phi_y}\right)^{c+1}$$

for all $\phi_y > 1$ and $c \geq 1$, and since

$$\left(\frac{2}{\Psi}\right) + \left(\frac{\phi_y}{\Psi} + \frac{1}{4}\right) = P(T_- = \tau_2 \text{ or } T_- = \tau_4) \leq 1,$$

we conclude that

$$E_\infty(c, \phi_x, \phi_y) < 1 - \left(\frac{1}{\phi_y}\right)^{c+1} = E_0(c, \phi_x, \phi_y).$$

This completes the proof of Theorem 3.

4. Proof of Theorem 4

As explained in the introduction, the first step to prove Theorem 4 is to identify a collection of events that simultaneously occur with positive probability and ensure that a given vertex, say the origin, dies before time one. These events are defined from the collection of independent Poisson processes introduced at the end of the model description as follows:

$$\begin{aligned} A_1 &= \{N_1^+(0) = 0 \text{ and } N_1^-(0) \geq (c+1)^2\} \\ A_2 &= \{N_1^+(z) = N_1^-(z) = 0 \text{ for all } z \in \mathbb{Z} \text{ such that } 0 < |z| \leq c+1\} \\ A_3 &= \{N_1^+(c+2) + \dots + N_1^+(c+n+1) \leq n \text{ for all } n > 0\} \\ A_4 &= \{N_1^+(-(c+2)) + \dots + N_1^+(-(c+n+1)) \leq n \text{ for all } n > 0\}. \end{aligned}$$

The times at which neighbors exchange a coin are unimportant in the proof of the theorem. Let A be the event that consists of the intersection of these four events.

Lemma 9 – For all $\mu \in [0, \infty]$, we have $P(\xi_1(0) = -1 \mid A) = 1$.

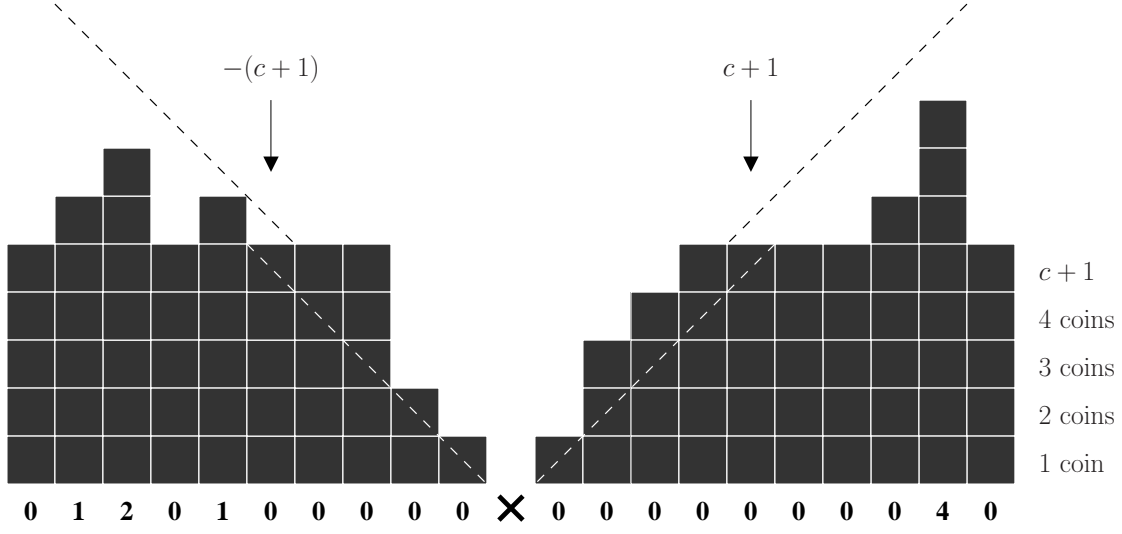


FIGURE 2. Typical configuration at time one when A occurs: the agent at 0 is dead and the fortune of the agents at distance at least $c+2$ from the origin is below the black dashed line. The numbers at the bottom of the picture give the number of coins these agents earned by time one. In the picture, we assume that these agents do not spend any coin, in which case the fortune of the agents within distance $c+1$ of the origin is above the white dashed line.

Proof. To begin with, we ignore the exchange of money between $c+1$ and its right neighbor and between $-(c+1)$ and its left neighbor. Recalling that an agent can receive one coin from a neighbor only if this neighbor has at least two more coins, on the event $A_1 \cap A_2$,

$$\xi_1(0) = -1 \quad \text{and} \quad c \geq \xi_t(z) \geq |z| - 1 \quad \text{for all} \quad 0 < |z| \leq c+1 \quad \text{and} \quad t \in (0, 1). \quad (7)$$

Note that the second inequality above becomes an equality when $\mu = \infty$. In this case, the total loss of coins among the $2c+3$ vertices around zero is given by

$$(c+1) + 2c + 2(c-1) + \cdots + 2 \times 1 + 2 \times 0 = (c+1)^2,$$

which explains our definition of the event A_1 . Observe that (7) implies that there are exactly c coins at vertex $c+1$ until time one. In particular, looking at the full system and allowing the exchange of money between $c+1$ and its right neighbor, on the event A_3 ,

$$\begin{aligned} & \text{number of coins traveling } c+1 \rightarrow c+2 \text{ by time one} \\ & \geq \text{number of coins traveling } c+2 \rightarrow c+1 \text{ by time one.} \end{aligned} \quad (8)$$

By symmetry, on the event A_4 ,

$$\begin{aligned} & \text{number of coins traveling } -(c+1) \rightarrow -(c+2) \text{ by time one} \\ & \geq \text{number of coins traveling } -(c+2) \rightarrow -(c+1) \text{ by time one.} \end{aligned} \quad (9)$$

Combining (7)–(9), we deduce that given the event A we must have $\xi_1(0) = -1$. \square

To prove that the event A has a positive probability, we let

$$\epsilon = -(1/2)(E(\phi) - 1) > 0 \quad \text{so that} \quad E(\phi) = 1 - 2\epsilon$$

and call vertex $z \in \mathbb{Z}$

$$\begin{aligned} \text{a right } \epsilon\text{-sink} & \quad \text{when} \quad \phi_z + \phi_{z+1} + \cdots + \phi_{z+n} \leq (n+1)(1-\epsilon) \quad \text{for all } n \in \mathbb{N} \\ \text{a left } \epsilon\text{-sink} & \quad \text{when} \quad \phi_z + \phi_{z-1} + \cdots + \phi_{z-n} \leq (n+1)(1-\epsilon) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Then, we have the following result.

Lemma 10 – We have $P(z \text{ is a right } \epsilon\text{-sink}) = P(z \text{ is a left } \epsilon\text{-sink}) = a > 0$.

Proof. Define the process

$$X_n = X_n(z) = \phi_z + \phi_{z+1} + \cdots + \phi_{z+n} - (n+1)(1-\epsilon) \quad \text{for all } n \in \mathbb{N}.$$

Since the random variables $\phi_z, \phi_{z+1}, \dots, \phi_{z+n}$ are independent and identically distributed, it follows from the strong law of large numbers that

$$\lim_{n \rightarrow \infty} \frac{X_n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n (\phi_{z+i} - (1-\epsilon)) = E(\phi) - (1-\epsilon) = -\epsilon < 0.$$

In particular, there exists N , fixed from now on, such that

$$P(X_n \leq 0 \text{ for all } n \geq N) = P\left(\sum_{i=1}^n (\phi_{z+i} - (1-\epsilon)) \leq 0 \text{ for all } n \geq N\right) \geq 1/2. \quad (10)$$

In addition, since $E(\phi) < 1-\epsilon$, we have $p = P(\phi \leq 1-\epsilon) > 0$ so

$$P(X_n \leq 0 \text{ for all } n < N) \geq P(\phi_{z+i} \leq 1-\epsilon \text{ for all } i < N) = p^N > 0. \quad (11)$$

Finally, combining (10) and (11) and using that the events $\{X_n \leq 0\}$ for different values of $n \in \mathbb{N}$ are positively correlated, we conclude that

$$\begin{aligned} P(z \text{ is a right } \epsilon\text{-sink}) &= P(X_n \leq 0 \text{ for all } n \geq 0) \\ &= P(X_n \leq 0 \text{ for all } n \geq N \mid X_n \leq 0 \text{ for all } n < N) P(X_n \leq 0 \text{ for all } n < N) \\ &\geq P(X_n \leq 0 \text{ for all } n \geq N) P(X_n \leq 0 \text{ for all } n < N) \geq (1/2) p^N > 0. \end{aligned}$$

It also follows from obvious symmetry that the probability that z is a left ϵ -sink is equal to the probability that it is a right ϵ -sink. This completes the proof. \square

Using the previous lemma, we can now prove that the event A has positive probability.

Lemma 11 – We have $P(A) > 0$.

Proof. Since the Poisson processes in the graphical representation are independent

$$P(A) = P(A_1) P(A_2) P(A_3) P(A_4).$$

In addition, for any given c finite, the first two events have positive probability while, by symmetry, the last two events have the same probability, i.e.,

$$P(A_1) P(A_2) > 0 \quad \text{and} \quad P(A_3) = P(A_4). \quad (12)$$

In particular, to conclude, it suffices to prove that the event A_3 has a positive probability. By conditioning on the event that vertex $c + 2$ is a right ϵ -sink, we get

$$\begin{aligned} P(A_3) &\geq P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) P(c + 2 \text{ is a right } \epsilon\text{-sink}) \\ &= a P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) \end{aligned} \quad (13)$$

where $a > 0$ according to Lemma 10. Now, let

$$Y_n = \text{Poisson}(n(1 - \epsilon)) \text{ be independent for all } n > 0.$$

Using that the events that define the event A_3 are positively correlated and recalling the definition of right ϵ -sink, we deduce that

$$P(A_3 \mid c + 2 \text{ is a right } \epsilon\text{-sink}) \geq P(Y_n \leq n \text{ for all } n > 0) = \prod_{n>0} P(Y_n \leq n). \quad (14)$$

In other respects,

$$\begin{aligned} \prod_{n>0} P(Y_n \leq n) &> 0 \quad \text{if and only if} \quad \sum_{n>0} -\log(1 - P(Y_n > n)) < \infty \\ &\quad \text{if and only if} \quad \sum_{n>0} P(Y_n > n) < \infty \end{aligned} \quad (15)$$

which follows from standard large deviations estimates for the Poisson distribution. Combining (13)–(15), we deduce that $P(A_3) > 0$ which, together with (12), gives the lemma. \square

Since the random variables ϕ_z are independent and identically distributed, we may apply the ergodic theorem together with Lemmas 9 and 11 to deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbf{1}\{\xi_1(z) = -1\} \geq P(A) > 0. \quad (16)$$

Note however that this does not imply our theorem since the probability of $A_1 \cap A_2$, and therefore the lower bound $P(A)$, depends on c , the initial number of coins per vertex.

The second step of the proof is to identify an infinite collection of vertices, that we call ϵ -sinks, that are removed eventually. The density of such vertices is bounded from below by a positive constant that does not depend on c . More precisely, we call vertex $z \in \mathbb{Z}$ an ϵ -sink if

$$\phi_{z-m} + \phi_{z-m+1} + \cdots + \phi_{z+n} \leq (m+n+1)(1 - \epsilon) \quad \text{for all } m, n \in \mathbb{N}. \quad (17)$$

Lemma 12 – We have $P(z \text{ is an } \epsilon\text{-sink}) \geq a^2 > 0$.

Proof. Let $A_{m,n}$ be the event in (17) and observe that

$$A_{m,0} \cap A_{0,n} \subset A_{m,n} \quad \text{for all } m, n \in \mathbb{N}.$$

In particular, the event that z is an ϵ -sink is

$$\bigcap_{m,n} A_{m,n} = \bigcap_{m,n} (A_{m,0} \cap A_{0,n}) = \left(\bigcap_m A_{m,0} \right) \cap \left(\bigcap_n A_{0,n} \right). \quad (18)$$

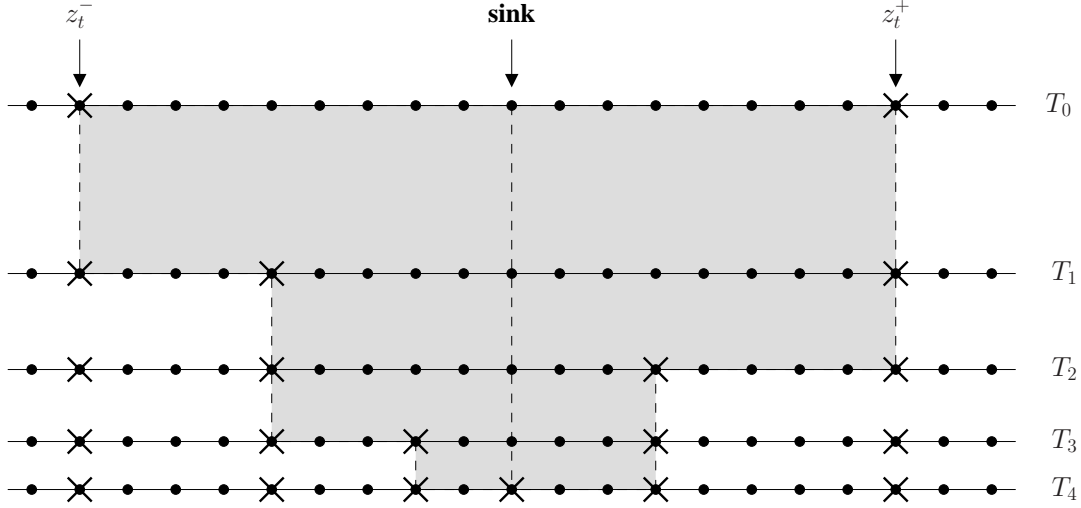


FIGURE 3. Picture of the construction in Lemma 13 with the sequence of stopping times T_i . The crosses \times represent the agents that are dead. The gray region shows the interval I_t from time $T_0 = 1$ until the sink dies. In our example, it takes four steps to kill the sink located at the center of the picture.

Using that $A_{0,n} = \{X_n \leq 0\}$ where the process (X_n) has been defined in the proof of Lemma 10 and obvious symmetry, we also have

$$P\left(\bigcap_m A_{m,0}\right) = P\left(\bigcap_n A_{0,n}\right) = P(X_n \leq 0 \text{ for all } n \geq 0) = a > 0 \quad (19)$$

according to Lemma 10. Combining (18) and (19), and using that the events $A_{m,0}$ and $A_{0,n}$ are positively correlated, we conclude that

$$P(z \text{ is an } \epsilon\text{-sink}) = P\left(\bigcap_{m,n} A_{m,n}\right) \geq P\left(\bigcap_m A_{m,0}\right) P\left(\bigcap_n A_{0,n}\right) = a^2 > 0.$$

This completes the proof. \square

To complete the proof of the theorem, the last step is to show that all the ϵ -sinks die eventually with probability one, which is done in the following lemma.

Lemma 13 – Assume that $x \in \mathbb{Z}$ is an ϵ -sink. Then $\xi_t(x) = -1$ for some t .

Proof. For all times t , we define

$$z_t^- = \sup \{z \leq x : \xi_t(z) = -1\} \quad \text{and} \quad z_t^+ = \inf \{z \geq x : \xi_t(z) = -1\}.$$

In view of (16) and since -1 is an absorbing state for each vertex,

$$I_t = (z_t^-, z_t^+) \text{ is bounded at time } t = 1 \text{ and nonincreasing in } t$$

for the inclusion. Now, set $T_0 = 1$ and define recursively

$$\begin{aligned} T_i &= \inf \{t > T_{i-1} : I_t \neq I_{t-}\} & \text{when } T_{i-1} < \infty \\ &= \infty & \text{when } T_{i-1} = \infty \end{aligned}$$

for all $i > 0$. See Figure 3 for a picture. Given that time T_i is finite and that the interval I_{T_i} is nonempty, by the definition of ϵ -sink, between time T_i and time T_{i+1} , the process

$$Z_t = \xi_t(z_{T_i}^- + 1) + \xi_t(z_{T_i}^- + 2) + \cdots + \xi_t(z_{T_i}^+ - 1)$$

is dominated stochastically by a one-dimensional random walk with a negative drift. This implies that the expected number of coins in the interval I_t is decreasing, therefore one of the vertices in the interval must reach state -1 in a finite time and

$$P(T_{i+1} < \infty \mid T_i < \infty \text{ and } I_{T_i} \neq \emptyset) = 1.$$

Recall also that the interval is bounded at time one and observe that, by definition of the stopping times, the length of the interval decreases by at least one at each step, i.e.,

$$|I_{T_0}| < \infty \quad \text{and} \quad |I_{T_{i+1}}| \leq |I_{T_i}| - 1 \quad \text{when} \quad T_i < T_{i+1} < \infty.$$

In summary, it takes only a finite number steps for I_t to become empty and the duration of each step is almost surely finite. Since in addition the sink dies at the time I_t becomes empty,

$$\inf \{t : \xi_t(x) = -1\} = \inf \{t : I_t = \emptyset\} < \infty$$

with probability one. This completes the proof. \square

As previously, since the random variables ϕ_z are independent and identically distributed, we may apply the ergodic theorem which, together with Lemmas 12 and 13, implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbf{1}\{\xi_t(z) = -1 \text{ for some } t\} \\ \geq \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{z=-n}^n \mathbf{1}\{z \text{ is an } \epsilon\text{-sink}\} \geq a^2 > 0. \end{aligned}$$

Since a does not depend on c , this proves Theorem 4.

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